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# Transient Response of a Current Pulse Propagating along a Vertical Channel Above Earth

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TRANSIENT RESPONSE OF A CURRENT PULSE  
PROPAGATING ALONG A VERTICAL CHANNEL ABOVE EARTH

by

Hussain A. Haddad and David C. Chang

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## ABSTRACT

In this report we investigate the electromagnetic response of a fast current pulse (with a rise time about  $.5 \mu$  sec. and a pulse width about  $1 \mu$  sec. at  $.707$  max. amplitude) propagating along a vertical line current of height  $h$  above a finitely conducting earth.

It was shown that the distortion in the early time portion of the pulse was mainly due to the finite conductivity of earth. Changing the conductivity from  $.01$  mho/m to  $.001$  mho/m resulted in a larger distortion in the pulse shape. However, the induction field becomes dominant for a later time portion of the pulse and it decays rapidly as distance increases from the current source.

## INTRODUCTION

In an earlier report [1], we have found the electromagnetic response of a current pulse propagating along a vertical lightning channel above a conducting half-space (Fig. 1). The vertical electric field  $E_z$  is shown to consist of three terms:

- a radiation field [2] which has a wave form resembling the propagating current wave envelope in the lightning channel except it decays as a  $d^{-1}$  from the current source where  $d$  is the radial distance of the observation point from the source.
- an induction field; a field that decays faster than a  $d^{-1}$ .

Both of the above two contributions can be derived exactly in terms of the dipole source and its perfect-conducting image.

- a correction field due to the finite conductivity of earth. An analytical expression of this term can be obtained only if the refractive index of earth is sufficiently large.

Quoting the result given in [1] we can write the normalized electric field for the first two contributions as

$$(\eta c / I_0) t_0 E_z^{(1)}(t_0; t) = 60 \left\{ f(t - t_0) + \eta(\eta^2 - 1) t_0 \int_0^{t-t_0} \frac{(t-t') f(t') dt'}{[(t-t')^2 + (\eta^2 - 1) t_0]} \right\}^{3/2} \quad (1)$$

where  $t_0 = d/c$  is the retardation time of electromagnetic waves over a distance  $d$ ;  $c$  is the speed of light in free space;  $\eta = c/v$  where  $v$  is the velocity of the current pulse along the vertical line current.

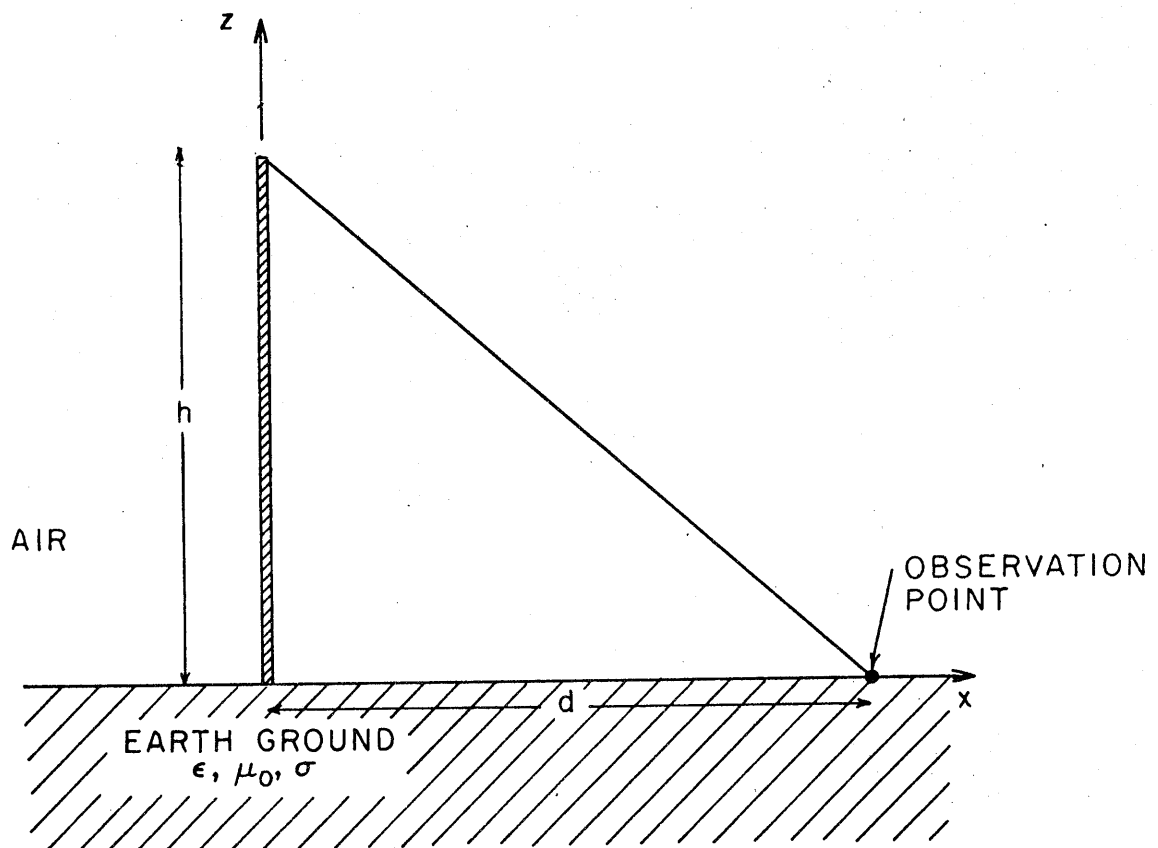


Figure 1. A vertical line current above a conducting half-space.

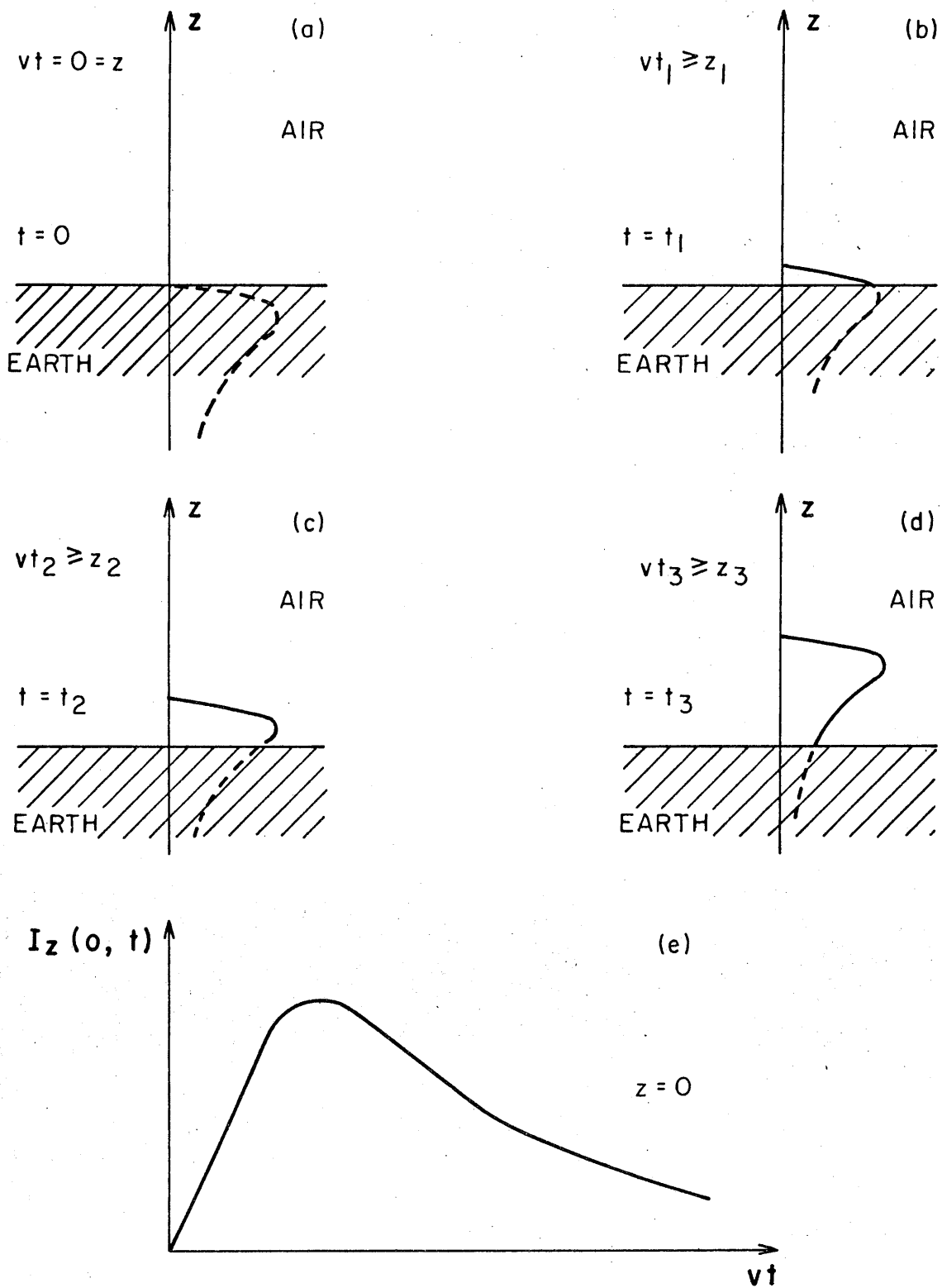


Figure 2 (a) - (4). A propagating current pulse at different times  $t$ .  
(e) A current pulse at  $z = 0$ .

Here  $f(t)$  is the shape of the current pulse so that at any given time the current distribution along the vertical channel is given as

$$I_z(z,t) = I_0 f(t - z/v) u(t - z/v) u(z) \quad (2)$$

and  $u(t')$  is the heaviside unit step function, i.e.  $u = 1$  for  $t' > 0$  and  $0$ , otherwise. In this report the wave function  $f(t)$  is chosen as

$$f(t) = e^{-p_1 t} - e^{-p_2 t} \quad (3)$$

and  $p_1, p_2$  are two parameters related to the rise time and the decay time of a current pulse. Figures 2(a)-(d) illustrate the observed current pulse for different times  $t$  for the current pulse given in (2).

Figure 2(e) shows that the current wave shape is defined at the base of the vertical line current. It is clear from these illustrations that the model we have used represents a current wave propagating along a vertical channel.

The ground correction which is the contribution due to the earth conductivity being finite is given as

$$(\eta c / I_0) t_0 E_z^{(2)}(t_0; t) = C_z(t_0; t) \quad (4)$$

where  $C_z(t_0; t)$  is expressed in terms of three individual response integrals as

$$C_z(t_0; t) = 60 \int_0^{t-t_0} [\partial f(t') / \partial t'] [\eta Q_1(t_0; t-t') - Q_2(t_0; t-t') - \eta^2 Q_3(t_0; t-t')] dt' \quad (5)$$

where

$$Q_1(t_0; t) = \epsilon_r^{-\frac{1}{2}} e^{-\Delta t_0 (u-1)/2} I_0 (\Delta t_0 [u-1]/2) \quad (6)$$

and

$$u = t/t_o$$

$$\Delta = \sigma/(\epsilon_r \epsilon_o)$$

where

$\sigma$  is the conductivity of earth.

$\epsilon_r$  is the relative permittivity of earth.

$\epsilon_o$  is the free space permittivity.

$I_o$  is the modified Bessel function of zero order.

Now  $Q_2(t_o; t)$  is given as

$$Q_2(t_o; t) = \partial_u \{ [\pi \Omega^2 u (1+\beta)]^{-\frac{1}{2}} \text{erf}(\alpha) K(v) \} \quad (7)$$

where  $\partial_u$  represents a derivative with respect to  $u$

$$\beta = (1 - 1/u^2)^{\frac{1}{2}}$$

$$v = 2\beta/(1+\beta)$$

$$\Omega = (t_o \sigma \mu_o c^2 / 2)^{\frac{1}{2}}$$

$\mu_o$  is the permeability in free space.

$K(v)$  is the complete elliptic integral of the first kind and  $\text{erf}(\alpha)$

is the error function.

The last term  $Q_3(t_o; t)$  can be given as

$$Q_3(t_o; t) = \partial_u [(\pi \Delta \epsilon_r t_o \tau)^{-\frac{1}{2}} F(\phi, M)] \quad (8)$$

$$\tau = (u + \eta^2 - 1)^{\frac{1}{2}}$$

$$\phi = 2 \tan^{-1} [(u-1)/\tau]^{\frac{1}{2}}$$

$$M = [(u + \tau)/2\tau]^{\frac{1}{2}}$$

where  $F$  is the elliptical integral of the first kind.



The total vertical electric field is then given as the sum of  $E_z^{(1)}$  and  $E_z^{(2)}$ . We note that, in the case of a lightning current pulse, it usually has a rise time of a few micro-seconds and a long decay time of tens or even hundreds of microseconds. The velocity of the pulse moving along the channel is also significantly lower than the speed of light (i.e.  $\eta^2 \gg 1$ ). Under these situations, the formula for  $C_z$  can be much simplified and computation based upon elliptical and Bessel functions can be eliminated [1]. In this report, however, we are mainly interested in those man-made pulses which vary much more rapidly than those of the lightning pulses. As a consequence, a more complete formula as given by (6) - (8) must be used. For convenience, the waveshape parameters  $p_1$  and  $p_2$  are chosen as  $1 \times 10^6 \text{ sec}^{-1}$  and  $3.5 \times 10^6 \text{ sec}^{-1}$ , respectively. The rise time of the pulse is then about  $.5 \mu \text{ sec.}$  and its width at  $.707$  of the maximum value is about  $1 \mu \text{ sec.}$

#### RESULTS AND CONCLUSION

A computer program has been devised to calculate  $(\eta c/I_0) t_0 E_z(t_0; t)$  which is the sum of the fields given in (1) and (4). Complete listing of the program is included in the Appendix. We have used a Romberg routine for the integration from 0 to  $t - t_0$  with an accuracy of the order  $1 \times 10^{-7}$ . Also two subroutines for the computation of the error function and the modified Bessel function have been developed with an accuracy in the order of  $1 \times 10^{-7}$ . Also a subroutine evaluating the elliptical integral (first and second kind) based upon a recurrence formula as given by [3] has been developed which proved to be extremely efficient for the prescribed accuracy. Using  $p_1 = 1.0 \times 10^6 \text{ sec}^{-1}$  and  $p_2 = 3.5 \times 10^6 \text{ sec}^{-1}$  in (3), the pulse will rise to a maximum value in  $.5 \mu \text{ sec.}$  and decays to  $e^{-1}$  of the maximum in

about 2  $\mu$  sec. The pulse shape  $f(t - t_0)$  has been shown in all of the next 8 figures as a dotted line.  $f(t - t_0)$  is called the radiation field and represents the undistorted pulse wave. However the distortion becomes evident as we add the induction and the correction field due to earth conductivity being finite.

Figures 3 and 4 shows the normalized electric field vs. time  $t - t_0$  for  $\sigma = .01$  mho/m and  $\sigma = .001$  mho/m respectively for different  $\eta$ 's (where  $\eta = c/v$  and  $v$  is the velocity of the current pulse propagating along the line current) at a fixed observation distance 1 km from the vertical line current. It is clear as the conductivity  $\sigma$  decreases from .01 mho/m (wet ground) to .001 mho/m (dry ground), the amplitude of the field in the early time portion of the pulse is distorted further from the undistorted case. For a slower current pulse corresponding to larger  $\eta$  values, influence of the induction field to the later portion of the response is indeed very evident.

Figures 5 and 6 are similar to Figures 3 and 4 except the observation distance  $d = 250$  meters. It is clear that the induction field should be more dominant in this case. But the effect of it diminishes rapidly as the velocity of the current pulse  $v$  increases.

In Figures 7 and 8,  $\eta$  is being fixed at 1.5 corresponding to a current wave propagating at a speed equal to two thirds of the speed of light, however, the pulse was shown at different distances from the base of the current channel ( $d = .25, .5, 1$  and  $3$  km.). It can be seen that at a later time and for a large distance, say,  $d = 3$  km, the tail portion of the response follows the undistorted waveshape closely, especially for  $\sigma = .01$  mho/m, but the early time portion of it is not. This is clear

since ground correction in (4) becomes dominant in this situation.

Figures 9 and 10 are similar to 7 and 8 except the propagating current pulse is slower,  $\eta = 3$ .

In conclusion, we have shown that the electromagnetic response due to a propagating current pulse along a vertical line current above a finitely conducting half-space can have a very different waveform from that of the waveform  $f(t - t_0)$ .

#### Reference

1. Chang, D.C., Haddad, H.A. and Maley, S.W., "Electromagnetic response of a current pulse propagating along a vertical lightning channel," Technical report no. 14, Oct. 1975, Electromagnetics Lab., Dept. of Electrical Engineering, University of Colorado, Boulder, Colo., 80309.
2. Uman, M.A., McLain, D.K., Fisher, R.J. and Krider, E.P., "Electric field intensity of the lightning return stroke," Journal of Geoph. Research, Vol. 78, No. 18, 3523-3529, June, 1973.
3. Abramovitz, M. and Stegun, I.A., Handbook of Mathematical Functions, page 599, U.S. Government Printing Offices, Washington, D.C., 1969.

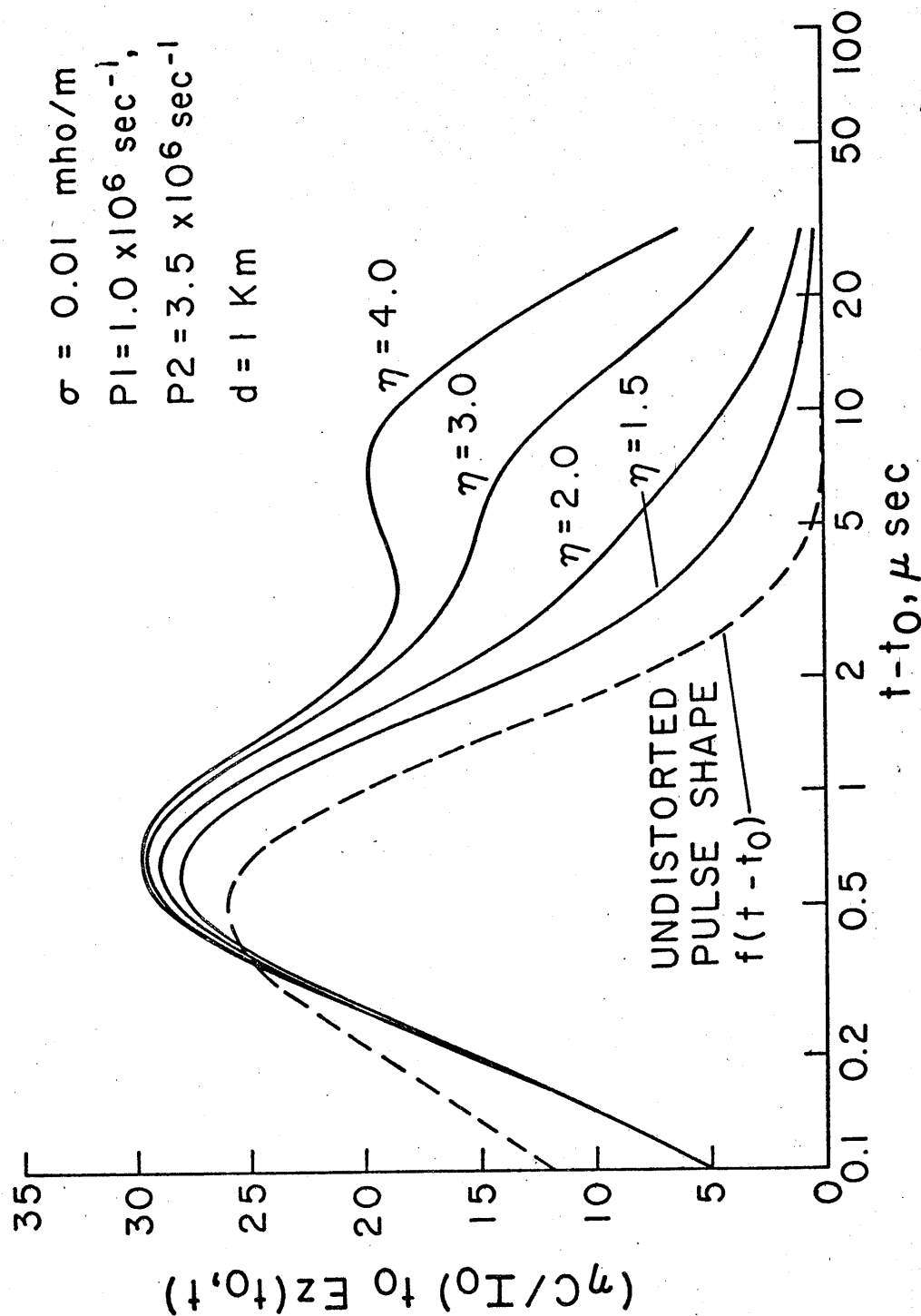


Figure 3. Normalized electric field versus time  $t = t_0$  at a fixed distance  $d(d = 1 \text{ Km})$  for different values of  $\eta$ . Earth conductivity  $\sigma = .01 \text{ mho/m}$  (wet ground).

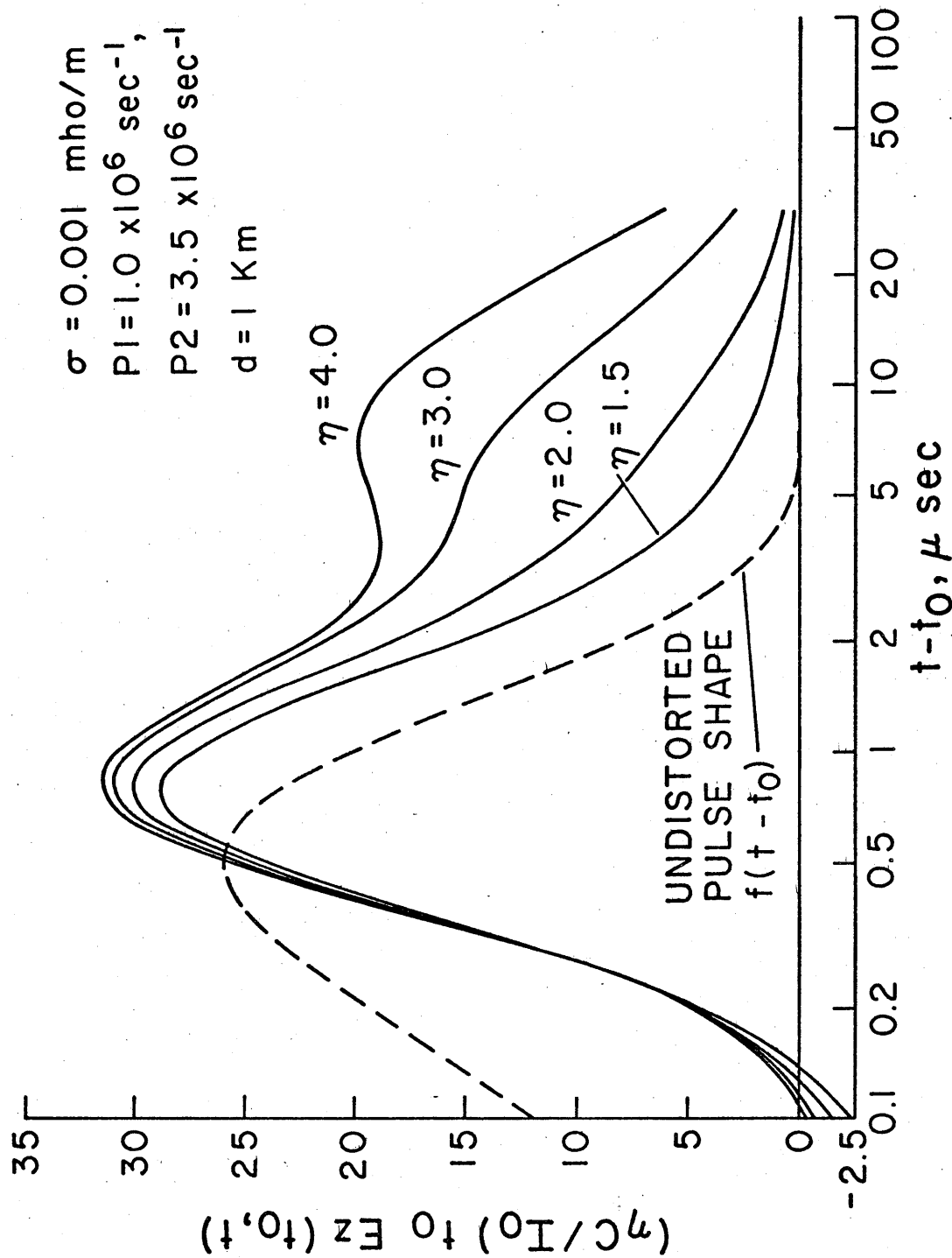


Figure 4. Normalized electric field versus time  $t - t_0$  at a fixed distance  $d$  ( $d = 1 \text{ Km}$ ) for different values of  $\eta$ . Earth conductivity  $\sigma = .001 \text{ mho/m}$  (dry ground).

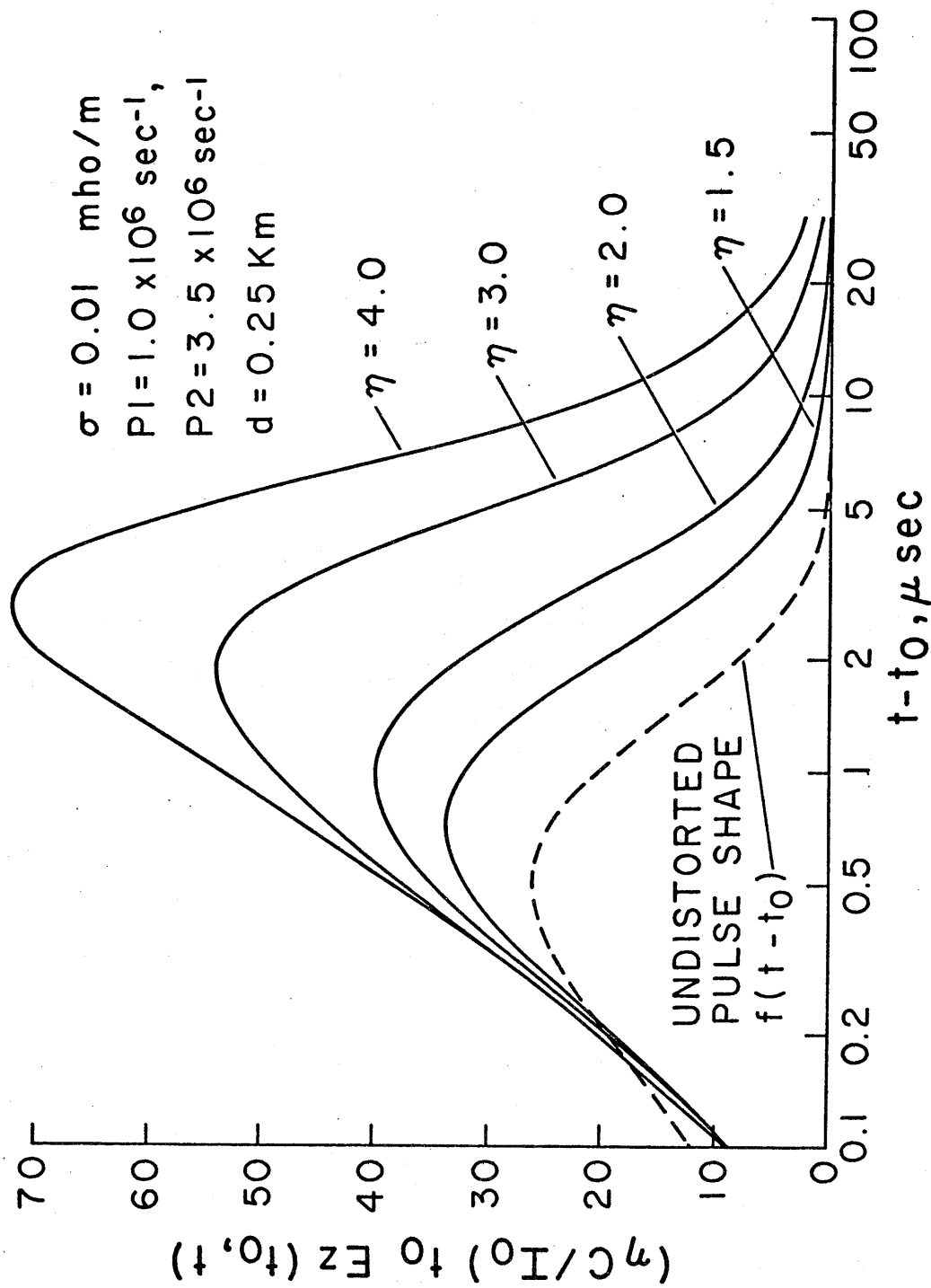


Figure 5. Normalized electric field versus time  $t - t_0$  at a fixed distance  $d(d = .25 \text{ Km})$  for different values of  $\eta$ . Earth conductivity  $\sigma = .01 \text{ mho/m}$  (wet ground).

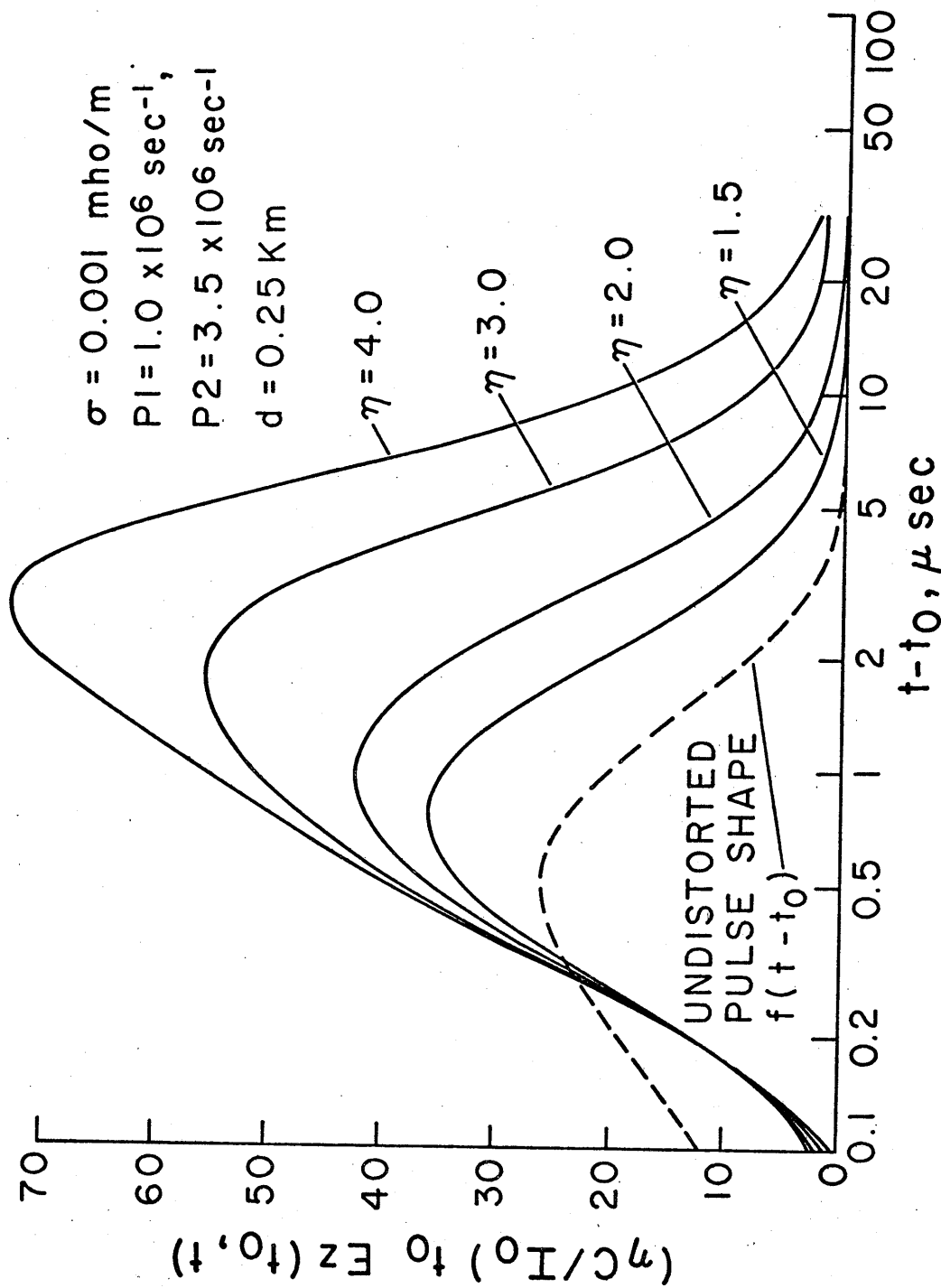


Figure 6. Normalized electric field versus time  $t - t_0$  at a fixed distance  $d$  ( $d = .25 \text{ Km}$ ) for different values of  $\eta$ . Earth conductivity  $\sigma = .001 \text{ mho/m}$  (dry ground).

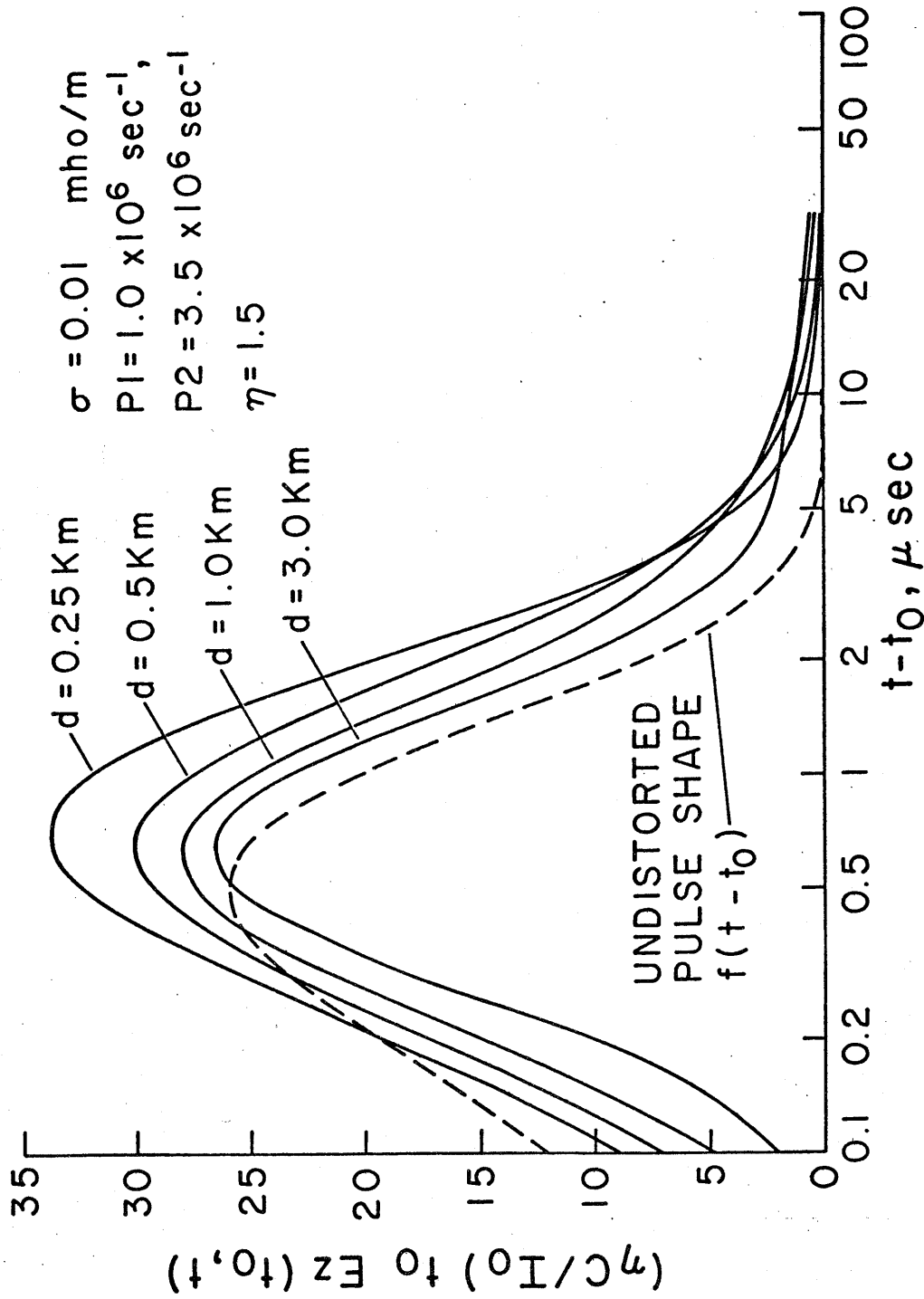


Figure 7. Normalized electric field versus time  $t - t_0$  for a fixed  $\eta$  ( $\eta = 1.5$ ) at different distances  $d$ . Earth conductivity  $\sigma = .01 \text{ mho/m}$  (wet ground).



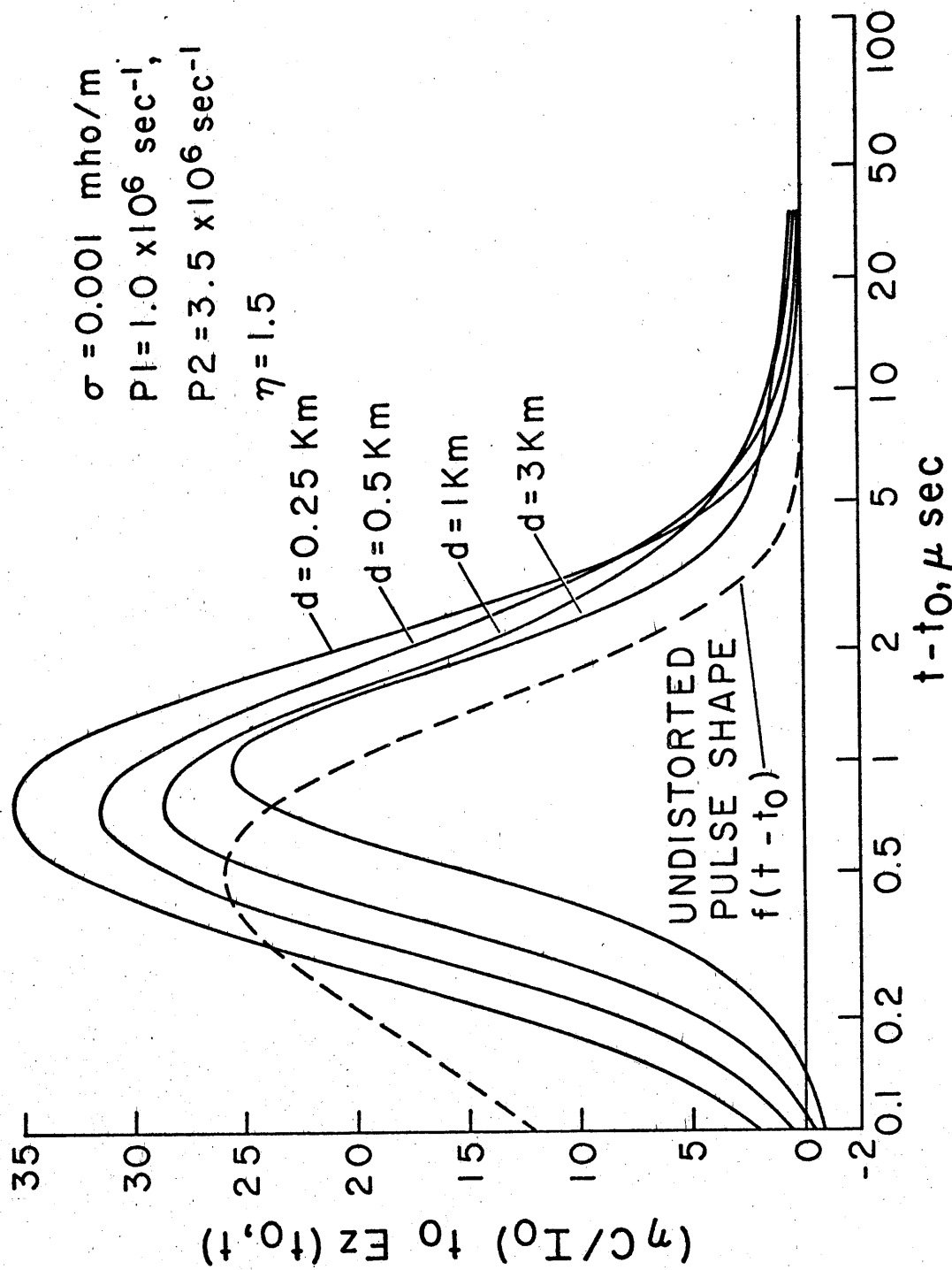


Figure 8. Normalized electric field versus time  $t - t_0$  for a fixed  $\eta$  ( $\eta = 1.5$ ) at different distances  $d$ . Earth conductivity  $\sigma = .001 \text{ mho/m}$  (dry ground).

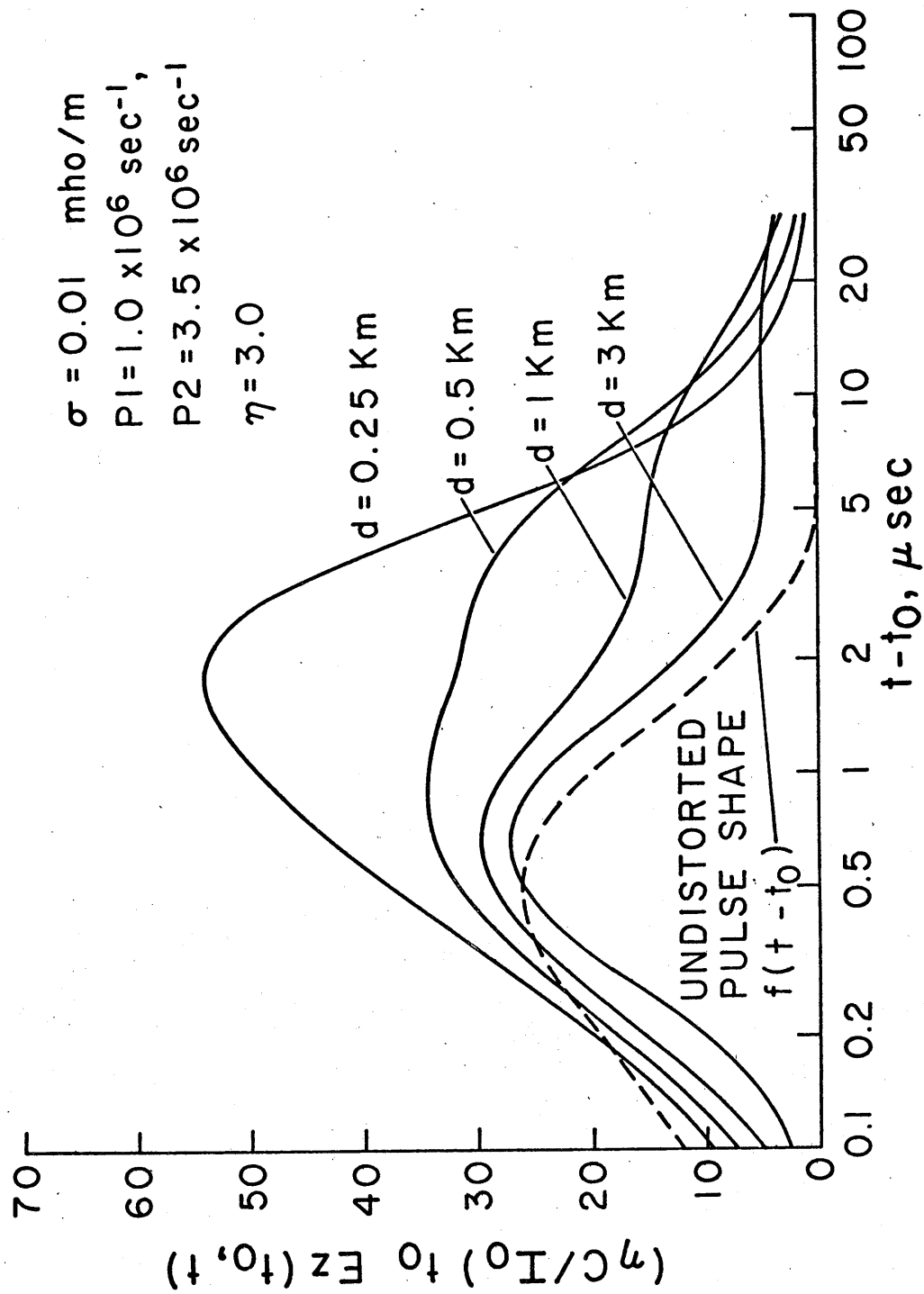


Figure 9. Normalized electric field versus time  $t - t_0$  for a fixed  $\eta$  ( $\eta = 3$ ) at different distances  $d$ . Earth conductivity  $\sigma = .01 \text{ mho/m}$  (wet ground).

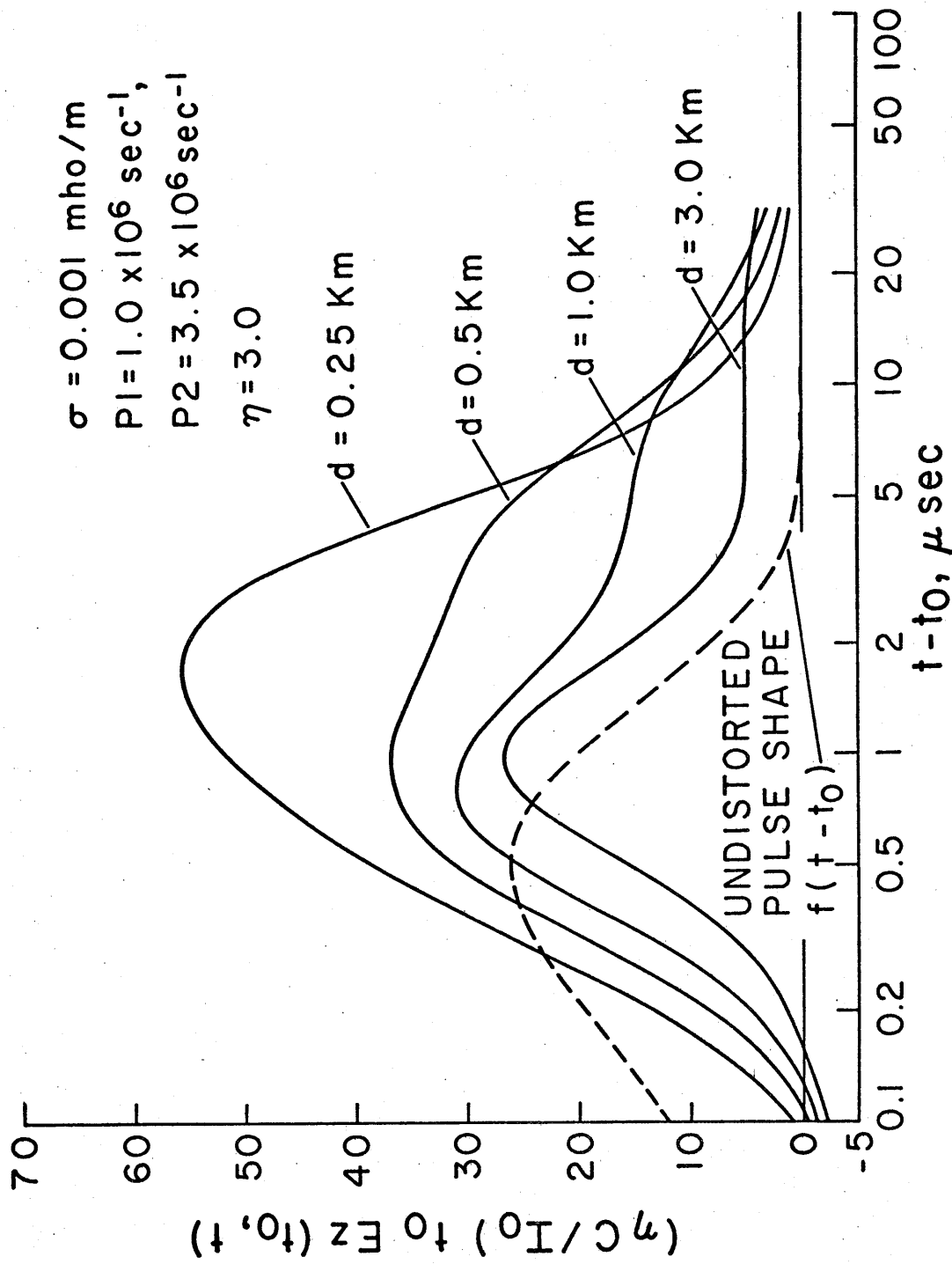
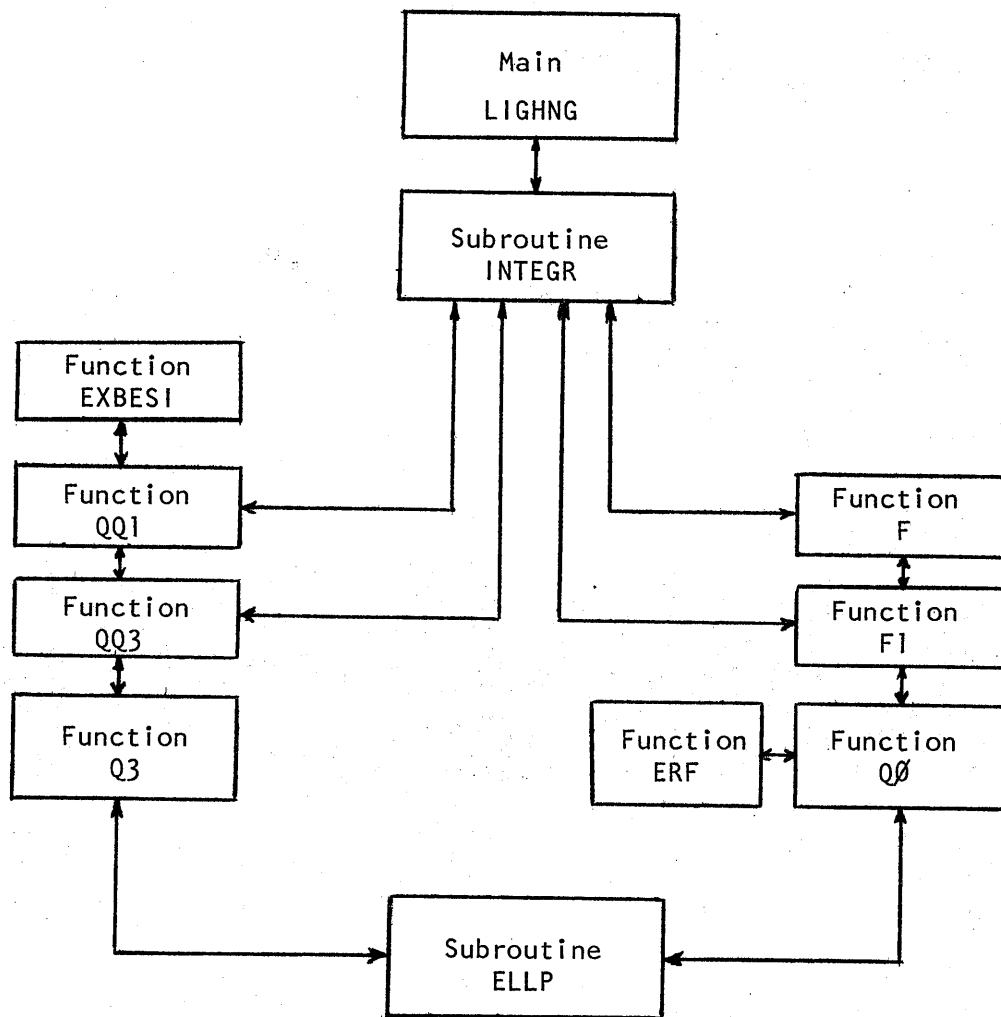


Figure 10. Normalized electric field versus time  $t - t_0$  for a fixed  $\eta$  ( $\eta = 3$ ) at different distances  $d$ : Earth conductivity  $\sigma = .001 \text{ mho/m}$  (dry ground).

## APPENDIX

### Computer Program Listings



A flow chart of the program to evaluate  
the response  $(\eta c/I_0)t_0 E_z(t_0; t)$

```

C      PROGRAM LIGHTING(INPUT,OUTPUT,PUNCH)
C      THIS PROGRAM IS USED TO STUDY THE PULSE PROPAGATION FOR A VERTICAL
C      LINE CURRENT,SUCH AS LIGHTNING FLASHES,ABOVE CONDUCTING EARTH,
C      ASSUMING A CURRENT PULSE OF THE FOLLOWING FORM
C       $I_0 * (\exp(-P_1 * T) - \exp(-P_2 * T)) * U(T)$ 
C      WHERE I0 IS SOME CONSTANT CURRENT TERM.
C      1/P1 AND 1/P2 ARE DECAY TIME CONSTANTS.
C      U(T) I0 A UNIT IMPULSE FUNCTION.
C      LOGICAL GT,GT1
C      1,GT2,GT3
C      COMMON PI,SIG,G(2),J,U,EA,DELTA,T0,EPSR
C      REAL K,MU0,NU
C      EXTERNAL F1, F, QQ3,QQ1
C      DIMENSION W2(2),WT2(2),W3(2),WT3(2)
C      DIMENSION W1(2),WT1(2)
C      N IS THE MAX. NUMBER OF ITERATION FOR THE INTEGRATION ROUTINE.
C      N=512
C      EPS IS THE REQUIRED ACCURACY IN THE INTEGRATION.
C      EPS=1.0E-7
C      PI=3.141592653
C      C IS SPEED OF LIGHT IN FREE SPACE.
C      C=2.99793E+8
C      EPSR IS THE RELATIVE PERMITTIVITY OF EARTH.
C      EPSR=10.
C      EPS0 IS THE PERMITTIVITY IN FREE SPACE.
C      EPS0=8.854E-12
C      COND IS THE CONDUCTIVITY OF GROUND.
C      COND=.001
C      PRINT 160,COND
C      160 FORMAT (1X,'#CONDUCTIVITY=#F9.5/')
C      DELTA=COND/EPSR/EPS0
C      MU0 IS THE PERMEABILITY OF FREE SPACE.
C      MU0=4.*PI*1.E-7

```

```

PRINT 41
41 FORMAT (1X,*I-T0 IS GIVEN IN MICRO-SECONDS*/)
      P(1)=P1 AND P(2)=P2
      P(1)=1.0E+6 $ P(2)=3.5E+6
      VELOCITY OF THE CURRENT IS V AND EA=C/V
      X IS THE OBSERVATION DISTANCE FROM THE LINE CURRENT.
      KK2 IS THE NUMBER OF CASES FOR EA AND X TO BE STUDIED.
      MM1 IS TYPICALLY 4.
      MM2 IS TYPICALLY 9.
      MM3 IS TYPICALLY 10.
      READ 5, KK2, MM1, MM2, MM3, TT0
      5 FORMAT (4I3, E7.1)
      DO 112 K2=1, KK2
      READ 4, EA, X
      4 FORMAT (F4.1, E7.1)
      T0=X/C
      PRINT 7, EA, T0
      7 FORMAT (1X, *EA= C/V= #F5.1/1X, *T0= #E13.5/)
      SIG=SQRT(T0*COND*MMU0*C*C/2.)
      DO 42 J1=1, 2
      G(J1)=T0*P(J1)
      WT(J1)=0. $ WT1(J1)=0.
      WT2(J1)=0. $ WT3(J1)=0.
      42 CONTINUE
      U2=TT0/T0
      A1=1.
      I1=1
      U1=1.
      PRINT 122
      122 FORMAT (1X, *NUMBER*, 5X* T-T0*, 7X* UNDISTORTED*, 5X* INDUCTION*, 3X
      1* GROUND CORRECT.*, 4X* GROUND CORRECT.*, 3X* TOTAL DIFF.*, 7X
      2* RESULTING*/22X* PULSE*, 27X* CONTAINS Q2*, 2X* CONTAINS Q1 AND Q3*,
      318X* PULSE*/)

```

```

DO 15 M1=1,MM1
DO 20 M2=1,MM2
DO 70 M3=1,MM3
U3=U2*(FLOAT(M2)+.1*FLOAT(M3-1))
  U=T/T0 WHERE T IS TIME
C
  U=U3+1.
  A=1./SQRT((U/EA)**2+1.-1./EA**2)
  AIN=0. S BIN=0.
  BIN1=0.
  BIN2=0. S BIN3=0.
  DO 90 J=1,2
    IF (U1.LE.1.)GO TO 22
    WT(J)=W(J)*EXP(G(J)*(U1-U))
    WT1(J)=W1(J)*EXP(G(J)*(U1-U))
    WT2(J)=W2(J)*EXP(G(J)*(U1-U))
    WT3(J)=W3(J)*EXP(G(J)*(U1-U))
22 CALL INTEGR(A1,A,EPS,N,F,VALUE,XT,GT)
    CALL INTEGR(U1,U,EPS,N, QQ3,EXTRA,XT2,GT2)
    CALL INTEGR(U1,U,EPS,N, QQ1,EXT,XT3,GT3)
    IF (GT2.OR.GT3) GO TO 55
    W(J)=VALUE+WT(J)
    W1(J)=RESULT+WT1(J)
    W2(J)=EXTRA+WT2(J)
    W3(J)=EXT +WT3(J)
    AIN=AIN-(-1.)**J*EXP(-G(J)*(U-1.))
    BIN=BIN+(EA*EA-1.)*(-1.)**J*W(J)
    BIN1=BIN1-(-1.)**J*G(J)*(Q0(U)-G(J)*W1(J))/2./PI
    BIN2=BIN2-(-1.)**J*( Q3(U)-G(J)*W2(J))*G(J)*EA*EA
    BIN3=BIN3+(-1.)**J*EA*G(J)*W3(J)
90 CONTINUE
  ST =U3*T0*1.E+6
  IF (ST-30.)26,92,92
26 R3 =BIN/AIN

```



```

C   AIN REPRESENTS THE UNDISTORTED PULSE SHAPE.
    AIN=AIN*60.
C   BIN REPRESENTS INDUCTION FIELD.
    BIN=BIN*60.
C   BIN1 REPRESENTS THE GROUND CORRECTION TERM THAT INCLUDES Q2.
    BIN1=BIN1*60.
C   BIN2 REPRESENTS THE GROUND CORRECTION TERM THAT INCLUDES Q3.
C   BIN3 REPRESENTS THE GROUND CORRECTION TERM THAT INCLUDES Q1.
C   THE INTEGRALS THAT CONTAIN Q1,Q2 AND Q3 ARE MENTIONED IN THE
C   TECHNICAL REPORT.
    R4=(BIN2+BIN3)*60.
    R1= (AIN+BIN+BIN1+R4 )
    R2=(BIN+BIN1+R4)
    PRINT 11,I1,ST,AIN,BIN,BIN1,R4,R2,R1
11  FORMAT (1X,I5,3XF9.4,6(3XE13.5)/)
    I1=I1+1
    U1=U
    A1=A
    IF (M2-6)70,21,21
70  CONTINUE
21  CONTINUE
20  CONTINUE
    U2=U2+10.
15  CONTINUE
92  CONTINUE
112 CONTINUE
    GO TO 88
55  PRINT 57,XT,XT1,XT2,XT3
57  FORMAT (1X,*ACTUAL ERRORS=*E15.7,3(3XE15.7)/)
88  STOP
    END

```

```

C      SUBROUTINE INTEGR (A,B,EPS,NSTEP,FCN,VALUE,X,G)
C      INTEGR IS A ROMBERG TYPE INTEGRATION ROUTINE
C      A=LOWER LIMIT,B=UPPER LIMIT,EPS=ACCURACY REQUIREMENT
C      NSTEP=MAX. NUMBER OF ITERATION,FCN=FUNCTION TO BE INTEGRATED
C      VALUE=RESULT OF INTEGRATION,G=LOGICAL STATEMENT IF FALSE THE
C      INTEGRATION IS COMPLETE OTHERWISE IT WILL RETURN ACTUAL ERROR X.
      LOGICAL G
      DIMENSION Q(16)
      H=B-A
      FCNA=FCN(A)
      FCNB=FCN(B)
      T=H*(FCNA+FCNB)/2.
      NX=1
      N=1
      1 K=2**N
      H=H/2.
      SUM=0.
      DO 2 I=1,NX
      XI=2.*FLOAT(I)-1.
      FCNXI=FCN(A+XI*H)
      2 SUM=SUM+FCNXI
      T=T/2.+H*SUM
      Q(N)=(T+H*SUM)*2.0/3.
      IF (N-2) 10,3,3
      3 F=4.
      DO 4 J=2,N
      I=N+1-J
      F=F*4.
      4 Q(I)=Q(I+1)+(Q(I+1)-Q(I))/(F-1.)
      IF (N-3) 9,5,5
      5 X=ABS(Q(1)-QX2) +ABS(QX2-QX1)
      COMP=X-3.*EPS
      IF (COMP) 11,11,8
      8 IF (NSTEP-K) 11,11,9

```

```

9  QX1=QX2
10 QX2=Q(1)
12 NX=NX*2
   N=N+1
   GO TO 1
11 VALUE=Q(1)
   G=NSTEP*LT.K
   RETURN
   END

```

```

C      FUNCTION ERF(AL)
C      THIS FUNCTION IS TO EVALUATE APPROX. THE ERROR FUNCTION.
C      THE INDUCED ERROR IS LESS THAN 3.*10**(-7).
      A1=.0705230784
      A2=.0422820123
      A3=.0092705272
      A4=.0001520143
      A5=.0002765672
      A6=.0000430638
      ERF=1.-(1.+((((A6*AL+A5)*AL+A4)*AL+A3)*AL+A2)*AL+A1)*AL)**(-16)
      RETURN
      END

```

```

C      FUNCTION EXBESI(DT)
C      EXBESI IS THE FUNCTION I0(X)*EXP(-X) WHERE I0 IS THE MODIFIED
C      BESSEL FUNCTION.
      T=DT/3.75
      IF (ABS(DT)).LE.3.75) GOTO 100
      A0=.39894228
      A1=.01328592
      A2=.00225319
      A3=.00157565
      A4=.00916281
      A5=.02057706
      A6=.02635537
      A7=.01647633
      A8=.00392377
      EXB =A0+A1*T**(-1)+A2*T**(-2)-A3*T**(-3)+A4*T**(-4)-A5*T**(-5)
      1+A6*T**(-6)-A7*T**(-7)+A8*T**(-8)
      EXBESI=EXB/SQRT(DT)
      RETURN
100  B2=3.5156229
      B4=3.0899424
      B6=1.2067492
      B8=.2659732
      B10=.0360768
      B12=.0045813
      Y=T**1
      EXB =1.+(((B12*Y+B10)*Y+B8)*Y+B6)*Y+B4)*Y+B2)*Y
      EXBESI=EXB*EXP(-DT)
      RETURN
      END

```

```

C      SUBROUTINE ELLP(PHI1,NU,F,E)
C      THE SUBROUTINE IS DESIGNATED TO EVALUATE THE ELLIPTIC INTEGRAL
C      OF THE FIRST (F) AND THE SECOND (E) KIND.
C      PHI1 IS THE UPPER LIMIT OF THE ELLIPTIC INTEGRAL.
C      NU IS THE MODULUS AND EQUOL K**2.
C      EF IS (F-E)/F
C      REAL    NU
C      DIMENSION A(50),B(50),C(50),PHI(50)
C      PI=3.141592653
C      IF (NU.NE.1.) GOTO 60
C      IF (PHI1.NE.PI/2.) GOTO 90
C      PRINT 95
C      95 FORMAT (1X,*F= INFINITY AND E=1.0*/ )
C      RETURN
C      90 F=ALOG(TAN(PI/4.+PHI1/2.))
C      E=SIN(PHI1)
C      RETURN
C      60 A(1)=1.
C      B(1)=SQRT(1.-NU)
C      C(1)=SQRT(NU)
C      PHI(1)=PHI1
C      M=0
C      E1=0.
C      E2=0.
C      DO 10 I=2,50
C      A(I)=.5*(A(I-1)+B(I-1))
C      B(I)=SQRT(A(I-1)*B(I-1))
C      C(I)=.5*(A(I-1)-B(I-1))
C      E1=.5*(I-1)*C(I)*C(I)+E1
C      IF (PHI1.EQ.PI/2.) GOTO 40
C      PHI(I)=PHI(I-1)+ATAN(B(I-1)*TAN(PHI(I-1)-M*PI)/A(I-1))+M*PI
C      50 IF (ABS(PHI(I)-M*PI).LT.PI/2.) GOTO 45
C      M=M+1
C      GOTO 50

```

```

45 E2=E2+C(I)*SIN(PHI(I))
40 IF (C(I).LT.1.E-8 ) GOTO 30
10 CONTINUE
20 PRINT 20
20 FORMAT (1X,'ELLP FAILED TO CONVERGE IN 50 ITERATION*')
30 IF (PHI(I).EQ.PI/2.) GOTO 80
F =PHI(I)/A(I)/2.** (I-1)
EF=.5*(C(I)*C(I)+E1)
E=F*(1.-EF)+E2
RETURN
80 F=PI/2./A(I)
EF=.5*(C(I)*C(I)+E1)
E=F*(1.-EF)
RETURN
END

```

```

FUNCTION Q0(U)
COMMON PI,SIG
REAL K,NU
AL=SIG*(U-1.)
BETA=SQRT(1.-1./U**2)
NU=2.*BETA/(1.+BETA)
CALL ELLP(PI/2.,NU,K,E)
Q0=2.*SQRT(PI/U)*ERF(AL)*K/SIG/SQRT(1.+BETA)
RETURN
END

```

```

FUNCTION F(X)
COMMON PI,SIG,G(2),J,U,EA
F=EXP(-G(J)*(U-SQRT((EA/X)**2-EA**2+1.)))
RETURN
END

```

C

```

FUNCTION F1(X)
F IS BEING USED IN THE INTEGRATION ROUTINE.
COMMON PI,SIG,G(2),J,U
F1=Q0(X)*EXP(G(J)*(X-U))
RETURN
END

```

```

FUNCTION Q3(UX)
COMMON PI,SIG,G(2),J,U,EA,DELTA,T0,EPSR
TAU=SQRT(UX*UX+EA*EA-1.)
PHAI=2.*ATAN(SQRT((UX-1.)/TAU))
ENU=(UX+TAU)/TAU/2.
CALL ELLP(PHAI,ENU,F,E)
Q3=F/SQRT(TAU*PI*DELTA*EPSR*T0)
RETURN
END

```

```
FUNCTION QQ1(X)
COMMON PI,SIG,G(2),J,U,EA,DELTA,T0,EPSR
DU=DELTA*T0*(X-1.)/2.
Q1=EXBESI(DU)/SQRT(EPSR)
QQ1=Q1*EXP(G(J)*(X-U))
RETURN
END
```

```
FUNCTION QQ3(X)
COMMON PI,SIG,G(2),J,U,EA,DELTA,T0,EPSR
QQ3=Q3(X)*EXP(G(J)*(X-U))
RETURN
END
```